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APPLICATION OF THE GRADIENT STRENGTH CRITERION AND THE BOUNDARY-ELEMENT METHOD TO A PLANE STRESS-CONCENTRATION PROBLEM

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A numerical algorithm for strength analysis of plane structural elements with stress concentrators is developed using the gradient strength criterion and the boundary-element method. As the first calculation test, the brittle fracture of a plate with a circular hole in tension is evaluated. To verify the algorithm and to compare the results of analysis with experimental data available in the literature, we consider symmetric and asymmetric problems of fracture of glass plates with a narrow elliptic hole in tension and compression. For all the problems, the accuracy of numerical results is estimated by comparison with analytical solutions. In comparison with the classical criteria, the use of the gradient strength criterion leads to better agreement between theoretical estimates and experimental data.

1. Formulation of the Problem. To answer the question of where and at which load a structural element with a stress concentrator begins to fail, we use a gradient approach: the nonuniformity of the stress state reduces the breaking ability of the stress in the region of maximum values, i.e., reduces its effectiveness. In [1, 2], Legan formulated a two-parameter gradient strength criterion. According to this criterion, to determine the breaking load, one should compare not the first principal stress σ_1 (used as the equivalent stress) but a certain effective stress $\sigma_e = \sigma_1/f(g_1, L_1, \beta)$, which is less than the equivalent stress, with the ultimate strength of the material σ_b. The denominator $f(g_1, L_1, \beta)$ is a function of the stress-field nonuniformity at the point of the body considered and two parameters that depend on the material properties. The effective stress is calculated by the formula

$$\sigma_e = \sigma_1/(1 - \beta + \sqrt{\beta^2 + L_1 g_1}). \tag{1.1}$$

The stress-state nonuniformity is characterized by the relative gradient of the first principal stress $g_1 = |\mathrm{grad}\,\sigma_1|/\sigma_1$ and is found from the elastic solution of the corresponding problem. The parameter L_1 has the dimension of length and it is determined from the condition that the gradient criterion agrees with linear fracture mechanics: $L_1 = (2/\pi)K_{\mathrm{I}c}^2/\sigma_b^2$, where $K_{\mathrm{I}c}$ is the critical coefficient of stress intensity. The dimensionless parameter β, which varies from 0 to 1, takes into account the quasi-brittle nature of fracture and it is calculated as the ultimate strength divided by the modulus of elasticity and the total strain for the moment of failure in uniaxial tension: $\beta = \sigma_b/(E\varepsilon_*)$. We assume that fracture begins at a point at the concentrator contour where the condition

$$\sigma_e = \sigma_b \tag{1.2}$$

is satisfied and propagates along the normal to the contour.

To apply the proposed strength criterion not only to demonstrative but also to practical problems, it is necessary to develop a numerical algorithm of strength analysis of structures with stress concentrators with the use of the most appropriate numerical method. In this paper, we use the boundary-element method (the

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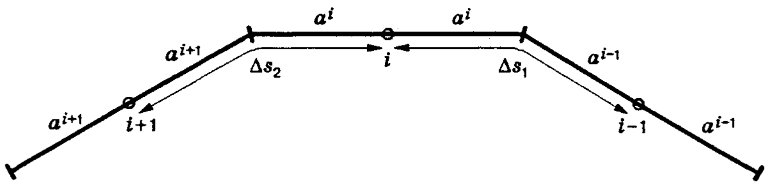

Fig. 1

method of fictitious loads). In this connection, the main difficulty is the need to determine not only the stress components, but also their derivatives with respect to spatial coordinates.

2. Scheme of Solution of the Problem. Main Problems. We outline a scheme of strength analysis of a plate with a stress concentrator with the use of the gradient strength criterion and the boundary-element method.

We confine ourselves to the case of loading the plate where the concentrator contour is free from tractions. In this case, one can use a tangential (circumferential) stress σ_t as the first principal stress σ_1 at the contour of the hole. Thus, for the problem considered, in the gradient strength criterion (1.1) and (1.2), we have

$$\sigma_1 = \sigma_t, \tag{2.1}$$

$$g_1 = |\mathrm{grad}\, \sigma_t|/\sigma_t. \tag{2.2}$$

The gradient of the tangential stress is calculated by the formula

$$|\mathrm{grad}\, \sigma_t| = \sqrt{(\partial\sigma_t/\partial s)^2 + (\partial\sigma_t/\partial n)^2},$$

where s and n are the tangent and the outward normal to the contour.

To apply the criterion (1.1), (1.2), it is necessary to solve the elastic problem of stress concentration and to determine the relative gradient of the tangential stress at any point at the contour of the concentrator. To solve the elastic problem, we use the boundary-element method in the form of the method of fictitious loads [3]. It should be noted that the boundary-element methods require discretization of only the boundary of the body, do not use approximations inside the domain, and lead to a fewer number of unknowns compared to the finite-element method; therefore, these methods are potentially more exact for stress-concentration problems. Better accuracy is achieved due to the fact that among the boundary-element methods, the method of fictitious loads uses an analytic expression for σ_t.

Using this method, one can determine the values of σ_t^i at the center of each ith boundary element at the contour of the concentrator and at the internal points of the plate. The derivatives of the tangential stress $\partial\sigma_t/\partial s$ and $\partial\sigma_t/\partial n$ at the contour, which are necessary for calculation of the gradient, are determined numerically by finite-difference formulas.

Substituting the values of σ_1 and g_1 calculated by (2.1) and (2.2) for each middle point of the boundary elements into expression (1.1) for σ_e and determining the point where the effective stress is maximum, we find the point where the fracture begins. The fracture occurs for the value of the applied load at which the effective stress attains the ultimate strength of the material, i.e., when condition (1.2) is satisfied.

Thus, the scheme of analysis is generally outlined. Now it is necessary to consider more carefully the question how to calculate the components of the gradient of σ_t.

To calculate the derivative $\partial\sigma_t/\partial s$ of the tangential stress along the tangent s to the contour of the concentrator, we use a three-point scheme of numerical differentiation with different steps (Fig. 1). The central derivative at the ith point at the contour is calculated from the formula

$$\frac{\partial\sigma_t^i}{\partial s} \approx \Big[\frac{\Delta s_1}{\Delta s_2}\sigma_t^{i+1} + \Big(\frac{\Delta s_2}{\Delta s_1} - \frac{\Delta s_1}{\Delta s_2}\Big)\sigma_t^i - \frac{\Delta s_2}{\Delta s_1}\sigma_t^{i-1}\Big]\Big/(\Delta s_1 + \Delta s_2),$$

where i, $i-1$, and $i+1$ denote the middle points of the boundary elements, σ_t^i, σ_t^{i-1}, and σ_t^{i+1} are the

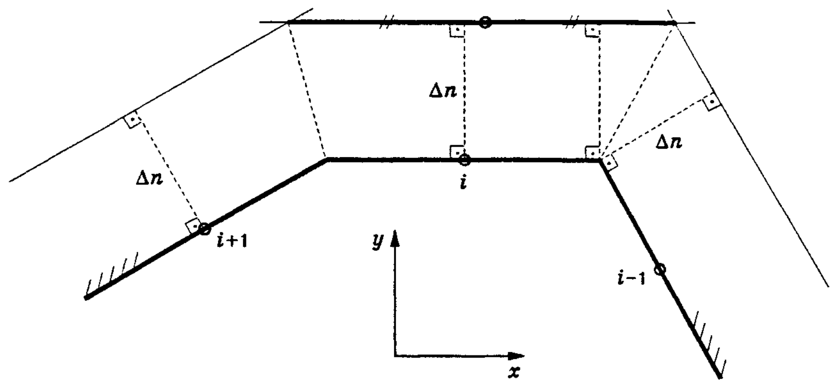

Fig. 2

corresponding values of σ_t obtained from the elastic solution, $\Delta s_1 = a^{i-1} + a^i$ and $\Delta s_2 = a^i + a^{i+1}$ are the absolute values of the increments of the arc coordinate s, i.e., the lengths of the corresponding arcs, and a^i is the half-length of the ith boundary element.

To calculate the derivative $\partial\sigma_t/\partial n$ of the tangential stress along the normal n to the contour of the concentrator, we use the simplest two-point scheme of numerical differentiation. The derivative for the ith point of the contour is calculated by the formula

$$\frac{\partial\sigma_t^i}{\partial n} \approx \frac{\sigma_t^i - \sigma_t^j}{\Delta n},$$

where σ_t^i and σ_t^j are the values of σ_t at the points i and j; the point i is the center of the ith boundary element and the point j lies inside the domain of the body and is at a distance Δn from the point i along the normal to the contour, which is taken the same for each point of the contour.

Precisely at this stage the following difficulty arises in calculations. Using the boundary-element method, one can calculate the stresses at the concentrator contour (at the middle points of the boundary elements) and at the internal points of the body. At first sight, the problem is solved. However, the stresses at the internal points can be computed, provided these points do not lie "too close" to the boundary. It has been found empirically that the solution is unreliable at the points inside a circle whose radius is equal to the length $2a$ of the element and whose center is at the middle point of the boundary element except for this middle point [3]. It should be added that the accuracy of the results even for internal points which satisfy the above condition and lie sufficiently far from the contour of the concentrator remains one or two orders of magnitude poorer than the accuracy of the results for the middle points of the boundary elements.

Consequently, one should develop a numerical algorithm that provides better accuracy in calculating stresses at the internal points of the body, in particular, in the neighborhood of the boundary.

3. Numerical Algorithm. We propose the following numerical algorithm. At a small distance Δn from the boundary elements at the contour of the concentrator (the basic contour), in the body we introduce a new boundary-element grid, which forms the auxiliary contour. Using the equations of equilibrium for an infinitesimal element that adjoins the contour of the concentrator, we formulate approximate boundary conditions for the auxiliary contour in terms of known values of the stresses at the basic contour and their derivatives with respect to the tangent to the contour. Applying the boundary-element method to the problem for the auxiliary contour and calculating the stresses at the middle of each boundary element, we actually determine the stresses at the internal points for the initial problem with higher accuracy.

Construction of an Auxiliary Contour. To construct an auxiliary contour, we draw straight lines parallel to each boundary element of the basic contour at distances Δn from them; intersections of the neighboring lines are assumed to be the ends of the elements of the auxiliary contour (if two lines coincide, which is the case where the initial boundary elements lie on the same line, we shift the common point of these elements a distance Δn along the normal); the middle points of the auxiliary elements are obtained by bisection (Fig. 2) of the corresponding segments.

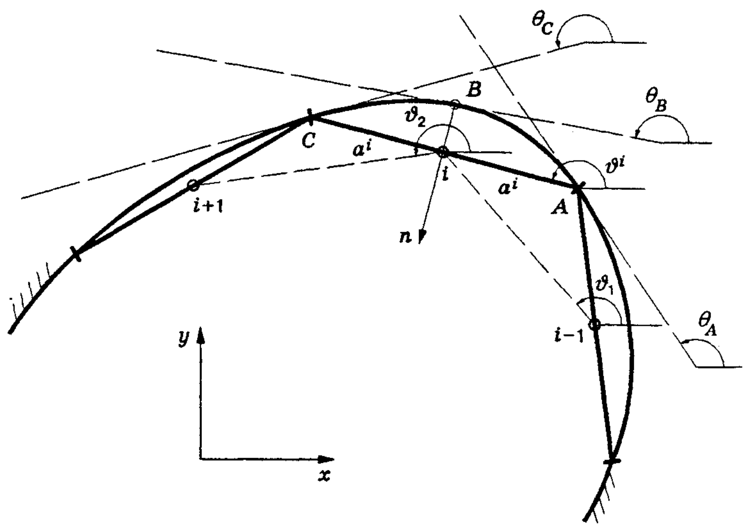

Fig. 3

Formulation of New Boundary Conditions. To derive the boundary conditions (σ_s and σ_n) on the auxiliary contour, we consider the stresses acting on an infinitesimal element at the boundary of the hole in the plate using the polar coordinate system (ρ, φ). Let the subscripts 1 and 2 refer to the basic and auxiliary contours, respectively.

The boundary of the hole in the plate is free from external loads:

$$\sigma_s\Big|_1 = 0, \qquad \sigma_n\Big|_1 = 0; \tag{3.1}$$

consequently,

$$\sigma_s\Big|_2 = \sigma_s\Big|_1 + \frac{\partial \sigma_s}{\partial \rho}\,d\rho = \frac{\partial \sigma_s}{\partial \rho}\,d\rho, \qquad \sigma_n\Big|_2 = \sigma_n\Big|_1 + \frac{\partial \sigma_n}{\partial \rho}\,d\rho = \frac{\partial \sigma_n}{\partial \rho}\,d\rho. \tag{3.2}$$

We express the derivatives in (3.2) via the known quantities. The differential equations of equilibrium of the element in the polar coordinates are written in the form

$$\sigma_n + \rho\,\frac{\partial \sigma_n}{\partial \rho} - \frac{\partial \sigma_s}{\partial \varphi} - \sigma_t = 0, \qquad \frac{\partial \sigma_t}{\partial \varphi} - \rho\,\frac{\partial \sigma_s}{\partial \rho} - 2\sigma_s = 0.$$

Hence, with allowance for (3.1), we find

$$\frac{\partial \sigma_n}{\partial \rho} = \frac{1}{\rho}\,\sigma_t, \qquad \frac{\partial \sigma_s}{\partial \rho} = \frac{1}{\rho}\,\frac{\partial \sigma_t}{\partial \varphi}.$$

Substituting the derivatives obtained into (3.2) and passing to the coordinate system (s, n) ($\Delta n = d\rho$ and $\partial s = \rho\,\partial\varphi$), we determine σ_s and σ_n at the contour 2 for the element considered:

$$\sigma_s\Big|_2 = \frac{\partial \sigma_t}{\partial s}\,\Delta n, \qquad \sigma_n\Big|_2 = \frac{1}{\rho}\,\sigma_t \Delta n. \tag{3.3}$$

Let $k = 1/\rho$ be the curvature of the contour 1 at the point considered. To determine the curvature, we use the numerical scheme shown in Fig. 3. Here the following notation is introduced: A and C are the first and the last points of the ith boundary element, respectively, B is the point of intersection of the actual contour of the concentrator and the normal n passing through the middle point of the ith boundary element, θ_A, θ_B, and θ_C are the angles between the x axis and the tangents to the actual contour of the concentrator at the points A, B, and C, respectively, ϑ^i is the slope angle of the ith boundary element, ϑ_1 and ϑ_2 are the angles between the x axis and the lines which pass through the middle points of the $(i-1)$th and the ith and the ith and the $(i+1)$th boundary elements, respectively, and k_1 and k_2 are the curvatures of the arcs AB and BC, respectively; $\psi_1 = \vartheta^i - \vartheta_1$ and $\psi_2 = \vartheta_2 - \vartheta^i$. Setting $\vartheta_1 = \theta_A$, $\vartheta^i = \theta_B$, $\vartheta_2 = \theta_C$, $k_1 = \psi_1/a^i$,

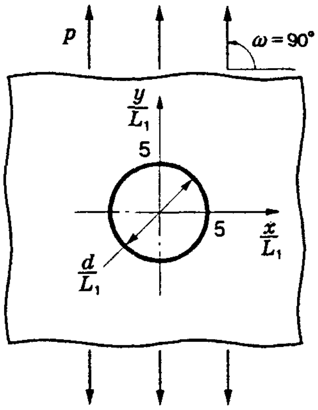

Fig. 4

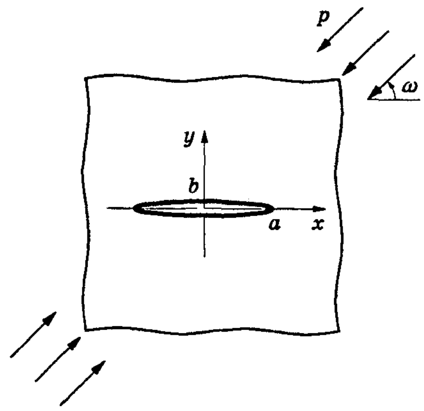

Fig. 5

and $k_2 = \psi_2/a^i$, we obtain an approximate formula for the curvature at the ith point of the contour of the concentrator $k^i \approx (k_1 + k_2)/2$.

Finally, the boundary conditions (3.3) for the ith boundary element at the auxiliary contour are written in the form

$$\sigma_s^i\Big|_2 = \frac{\partial \sigma_t^i}{\partial s}\Delta n, \qquad \sigma_n^i\Big|_2 = k^i \sigma_t^i \Delta n.$$

4. Examples of Analysis. Evaluation of the Results. As the first test problem, we considered a plate with a circular hole in tension. Moreover, to verify the algorithm and compare the results of analysis with the experimental data [4], the symmetric and asymmetric problems of failure of glass plates with a narrow elliptic hole in tension and compression were considered.

In all the problems, we set $\beta = 1$ (brittle fracture). Below, we give the results for five problems.

Circular Hole in the Plate. We consider Problem 1 of uniaxial tension of an infinite plate with a circular hole of diameter d (Fig. 4) for $d/L_1 = 10$. The parameter L_1 is equal to the critical dimension of a defect of the Griffith crack type [1].

A numerical solution was obtained for division of the boundary of the hole into 60 and 360 boundary elements. A boundary-element grid was constructed in such a manner that the middle points of the elements lie at the symmetry axes of the problem. Calculations were performed for $\Delta n/L_1 = 10^{-3}$. An increase or decrease in Δn by a factor of 10 did not affect the results.

Elliptic Hole in the Plate. We consider several problems of a plate with a narrow elliptic hole which is uniaxially stretched or compressed at infinity (Fig. 5):

— Tension at an angle $\omega = 90°$ with the major axis of the ellipse (Problem 2);
— Compression at an angle $\omega = 0$ with the major axis of the ellipse (Problem 3);
— Compression at an angle $\omega = 30°$ with the major axis of the ellipse (Problem 4);
— Compression at an angle $\omega = 45°$ (Problem 5);

For the purpose of comparison with the experimental data of [4], we approximate the parameter L_1 as follows. From the gradient strength criterion (1.1) and (1.2), for the moment of onset of fracture, we have the relation (for $\beta = 1$).

$$\sigma_b = \sigma_t^*/\sqrt{1 + L_1 g_1}, \tag{4.1}$$

where σ_t^* is the circumferential stress at the point where the fracture begins. This stress can be calculated by the nominal stress at the moment of onset of the fracture p^* and the concentration coefficient α at this point:

$$\sigma_t^* = p^* \alpha. \tag{4.2}$$

TABLE 1

No.		φ, deg	p^*/σ_b	α	g_1, mm^{-1}
1	Analytical solution	0	0.40369	3.0	0.46667
	Numerical solution (60 elements)	0	0.40304	2.9999	0.46191
	(error, %)	(0)	(0.16)	(0.003)	(1.02)
	Numerical solution (360 elements)	0	0.40367	3.0000	0.46654
	(error, %)	(0)	(0.005)	(0.000)	(0.03)
	Classical criterion	0	0.33333	3.0	—
2	Analytical solution	0	0.06667	21.0	32.246
	Numerical solution (360 elements)	0	0.06554	20.985	29.945
	(error, %)	(0)	(1.70)	(0.07)	(7.14)
	Numerical solution (600 elements)	0	0.06611	20.995	31.118
	(error, %)	(0)	(0.84)	(0.02)	(3.50)
	Experimental results	0	0.06667	—	—
	Classical criterion	0	0.04762	21.0	—
3	Analytical solution	0	−1.5812	−1.0	50.394
	Numerical solution (360 elements)	0	−1.5398	−0.99864	45.835
	(error, %)	(0)	(2.62)	(0.14)	(9.05)
	Numerical solution (600 elements)	0	−1.5613	−0.99955	48.219
	(error, %)	(0)	(1.26)	(0.05)	(4.32)
	Experimental results	0	−1.75	—	—
	Classical criterion	0	−1.0	−1.0	—
4	Analytical solution	59.144	−0.38856	−2.9510	10.575
	Numerical solution (360 elements)	60.441	−0.39208	−2.9103	10.143
	(error, %)	(2.19)	(0.91)	(1.38)	(4.09)
	Numerical solution (600 elements)	59.407	−0.39004	−2.9437	10.689
	(error, %)	(0.44)	(0.38)	(0.25)	(1.08)
	Experimental results	58.47	−0.55	—	—
	Classical criterion	55.087	−0.33257	−3.0069	—
5	Analytical solution	67.248	−0.40020	−2.7347	6.6446
	Numerical solution (360 elements)	66.580	−0.40282	−2.7498	7.6252
	(error, %)	(0.99)	(0.66)	(0.55)	(14.8)
	Numerical solution (600 elements)	66.909	−0.40148	−2.7428	7.1402
	(error, %)	(0.50)	(0.32)	(0.30)	(7.46)
	Experimental results	78.05	−0.57	—	—
	Classical criterion	64.645	−0.36141	−2.7670	—

Note. The experimental results were obtained by averaging the values over all the tests.

Substituting (4.2) into (4.1), after certain manipulations, we obtain the equation for L_1

$$L_1 = \left(\left(\alpha \frac{p^*}{\sigma_b}\right)^2 - 1\right) \Big/ g_1. \tag{4.3}$$

Upon tension of a glass plate at an angle $\omega = 90°$ with the major axis of the ellipse (Fig. 5), it was found experimentally [4, pp. 199, 200] that $p^*/\sigma_b = 1/15$. Fracture begins at the vertex of the ellipse, where the concentration coefficient is $\alpha = 1 + 2a/b = 21$. Here $a = 6.35$ mm and $b = 0.635$ mm are the major and minor semiaxes of the ellipse. The relative gradient g_1 at the vertex is calculated from the formula [1]: $g_1 = (\alpha - 1)^2(1 + 1/(2\alpha))/(2a) = 32.246$ mm^{-1}. Substituting all the values into (4.3), we find $L_1 = 0.029771$ mm.

The numerical solution of the problem was found by partitioning the boundary of the hole into 360 and 600 boundary elements. A boundary-element grid was constructed in such a manner that the middle points of the elements lie at the axes of the ellipse. Calculations were performed for $\Delta n = 0.001$ mm. A decrease in

Δn by a factor of 10 changes the results by no more than 0.12%.

Analysis of Results. In analyzing the results, we are interested in the following:

— the direction of fracture which is determined by the angle φ between the major axis of the ellipse and the normal to the contour at the point where the fracture begins;

— the ratio of the nominal breaking load and the ultimate strength of the material p^*/σ_b;

— the stress-concentration coefficient α at the point where the fracture begins;

— the relative gradient of the tangential stress g_1 at the point where fracture begins.

We compare the above-mentioned parameters, which were found numerically, with their values obtained from analytical solutions; we determine the relative (percent) errors of calculation of each value. The theoretical estimates which were found by the gradient strength criterion are also compared with the experimental data of [4]. Moreover, it is of interest to compare the experimental data with the results obtained by the classical strength criterion $\sigma_t^{\max} = \sigma_b$ and the analytical solutions. All the results are listed in Table 1.

A comparison of the numerical results with the analytical solutions shows that the calculation error for the limiting load for 360 boundary elements does not exceed 3%, which is admissible in view of the difference between the theoretical and experimental data. As is seen from Table 1, the calculation errors decrease with increase in the number of elements.

It is noteworthy that the theoretical results obtained with the use of the gradient strength criterion are closer to the experimental data than those obtained by means of the classical strength criterion $\sigma_t^{\max} = \sigma_b$.

The accuracy of calculation of the relative gradient of the circumferential stress is always poorer than that of this stress.

We also note that, in the asymmetric problems, the errors listed in Table 1 are mainly due to the fact that the middle point of any boundary element does not coincide with a point at the contour of the hole where, according to the analytical solution, fracture is expected to begin. For example, if the boundary-element grid in Problem 5 is constructed in such a manner that the direction of the normal to one of the 360 boundary elements coincides with the direction of fracture $\varphi = 67.248°$, which was previously found from the analytical solution, the numerical method gives the stress-concentration coefficient $\alpha = -2.7354$ and the relative gradient $g_1 = 7.1028\ \text{mm}^{-1}$. In comparison with the values given in Table 1, these results agree better with the analytical solution.

Conclusions. Using the gradient strength criterion and the boundary-element method, namely, the method of fictitious loads, we have developed a numerical algorithm for strength analysis of plane structural elements containing stress concentrators, which allows one to apply the gradient criterion to both demonstrative and practical problems with sufficient accuracy.

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